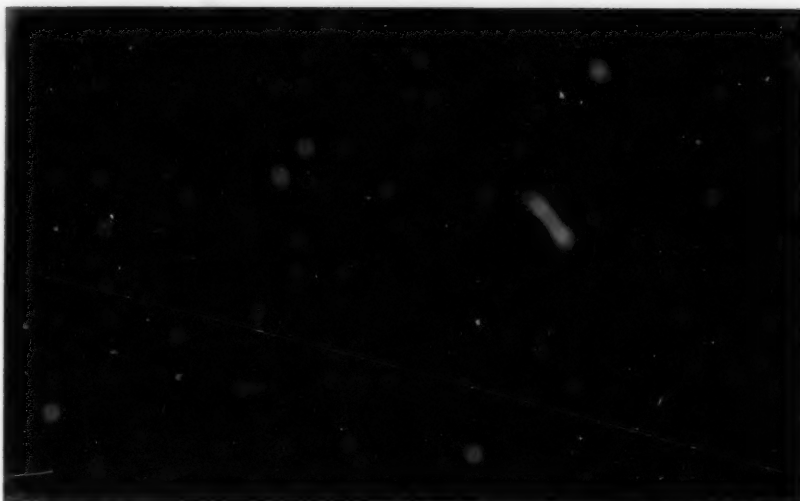




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**Implicit Enumeration of Hyperpaths
in Logit Models for Transit Networks**

by

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ABSTRACT

This paper investigates the application of multinomial logit distribution models to urban transit networks where every set of competitive transit lines is described by a particular graph structure called hyperpath. It shows that the standard multinomial logit model does not lead to a practical algorithmic implementation similar to that obtained for the private vehicle networks and should be replaced with a hierarchical logit. This latter model seems to transcend the inherent limitations of the simple logit model which are amplified by the hyperpath framework.

Key words : transit network, passenger assignment, oriented hypergraphs, active hyperpaths, logit models, implicit enumeration.

RÉSUMÉ

Cet article porte sur l'application des modèles de distribution "logit" multinomiaux aux réseaux de transports en commun dans lesquels chaque ensemble de lignes de transport compétitives est représenté par une structure de graphe particulière appelée hyperchemin. On montre que le modèle "logit" multinomial classique ne se prête pas dans ce cas à une implantation algorithmique efficace, contrairement au cas des réseaux de transport par automobile. On propose d'utiliser un modèle "logit" hiérarchique qui se révèle plus satisfaisant que le modèle classique, tant sur le plan de la modélisation que sur le plan algorithmique, dans un contexte d'hyperchemins.

Mots-clés : réseaux de transport en commun; affectation des passagers; hypergraphes orientés; hyperchemins actifs; modèles "logit"; énumération implicite.

1 Introduction

On a transit network with overlapping or competitive lines, daily commuters usually have a set of perceived equivalent itineraries between origin and destination pairs. For any given trip, the itinerary used depends on the arrival of the first carrier of the competitive lines at specific bus stops. For these networks, the elementary path concept is inadequate to model the passenger's itinerary choice. This leads to the introduction of the hyperpath concept in transit networks ([27]), which not only captures reasonably well the passenger's choice of a set of competitive lines at a bus stop, but also often allows a straightforward adaptation of existing private vehicles transport models to the transit network case (see, for instance, [25, 5, 34]).

The graph description of a transit network using the hyperpath concept may be viewed as a particular directed hypergraph—an F-graph—and transit network hyperpaths as F-paths ([13]). The seemingly disadvantage of using hyperpaths instead of elementary paths passenger assignment models stems from the fact that for the same transit network the number of hyperpaths is significantly higher than that of elementary paths. For instance, at a given stop served by k lines start k first-arc disjoint paths and $2^k - 1$ first-arc disjoint hyperpaths (i.e., one per non-empty subset of lines). Consequently, any problem formulated with hyperpaths is unpractical if an explicit enumeration of hyperpaths is required.

Practical implementations of existing transit network models, such as passenger assignment and origin-destination trip estimation models ([27, 25, 5]), depend on whether or not the corresponding solution procedure obviates the explicit enumeration of hyperpaths. This paper investigates the multinomial logit splitting of flows among a set of active hyperpaths connecting a given origin-destination pair. It shows that, in contrast to the vehicle assignment on directed graphs ([8, 29]), the standard logit model does not lead here to an efficient algorithmic implementation and it should be replaced with a hierarchical logit model (also referred to in the literature as structured or nested logit model). Full accounts of the hierarchical logit model and its application to travel demand forecasting are given in [33, 31, 19].

The paper is organized as follows. The next Section describes the modeling of a transit network by a particular directed hypergraph, and defines the

cost of a hyperpath. Section 3 examines different aspects of the flow splitting problem such as the diversion of a hyperpath flow among its paths and the definition of the set of active hyperpaths. An algorithm which blends the determination of active boarding arcs with the shortest hyperpath computation is described. Finally, Section 4 investigates the multinomial logit distribution models with emphasis on the computational efficiency. It shows that only the hierarchical logit model allows to distribute flow without an explicit enumeration of active hyperpaths.

2 Directed hypergraphs and transit networks

The support F-graph

Directed hypergraphs—a variant of Berge's hypergraphs ([2, 3])—or more precisely F-graphs ([13]) are particularly useful to the modeling of transit networks. A F-graph is a pair $\mathcal{H} = (N, E)$, where $N = \{v_1, v_2, \dots, v_n\}$ is the set of nodes and $E = \{e_1, e_2, \dots, e_m\}$ is the set of F-arcs. A F-arc $e = (t(e), h(e))$ is identified by its head $h(e) \subseteq N$ and its tail $t(e) \in N \setminus h(e)$. The size of a F-graph is a function of the cardinality of the F-arc heads:

$$size(\mathcal{H}) = \sum_{e \in E} (|h(e)| + 1).$$

Consider a F-arc $e = (t(e), h(e)) \in E$ then every

$$e' = (t(e'), h(e')) \quad \text{such that} \quad \begin{cases} h(e') \subseteq h(e) \\ t(e') = t(e). \end{cases}$$

is a *contained* F-arc. A *full* F-graph $\mathcal{H} = (N, E)$ is a F-graph which includes all the contained F-arcs. The *support* of a F-graph is obtained from the original F-graph by suppressing all strictly contained F-arcs. A full F-graph, such as that associated with a transit network, is therefore unambiguously described by its support F-graph. Note that the size of the support graph is considerably lower than that of the full graph. Indeed, if E is the set of support F-arcs, then the size of the associated full F-graph $\mathcal{H}'$ is

$$size(\mathcal{H}') = \sum_{e \in E} (|h(e)| + 2)2^{|h(e)|-1} - |E|.$$

An interesting subclass of F-graphs is the class of the acyclic F-graphs, for which it is possible to re-number all the nodes in such a way that for any

F-arc $e = (t(e), h(e))$ and for any $u \in h(e)$, $t(e) < u$ is verified (this order is often called a topological order).

In a F-graph, let $A(v) = \{e \in E \mid v = t(e)\}$ denote the set of F-arcs incident-out of v (the forward-star of node v), and similarly let $B(v) = \{e \in E \mid v \in h(e)\}$ denote the set of F-arcs incident-into v (the backward-star of node v). A path q_{xy} of length m connecting y and x in F-graph $\mathcal{H} = (N, E)$ is a sequence of nodes and F-arcs:

$$q_{xy} = (x = t(e_1), e_1, t(e_2), e_2, \dots, e_m, y),$$

where

$$t(e_{i+1}) \in h(e_i), \quad \text{for } i = 2, \dots, m-1, \quad \text{and } y \in h(e_m).$$

Node y is said to be connected to node x if there exists a path $q_{xy} \in \mathcal{H}$. A F-path p_{xy} of $\mathcal{H}$ is defined as a minimal acyclic sub-graph $\mathcal{H}_p = (N_p, E_p)$ such that:

$$E_p \subseteq E, \quad x, y \in N_p = \bigcup_{e \in E_p} h(e),$$

$$\text{and } i \in N_p \Rightarrow \exists q_{iy} \in \mathcal{H}_p.$$

Node y is F-connected to node x if there exists a F-path $p_{xy} \in \mathcal{H}$. In a weighted F-graph each F-arc e is assigned a cost $c(e)$ and a set of non-negative node-arc coefficients $a(e, i)$, for each $i \in h(e)$. When a support F-graph is used instead of the corresponding full graph, it will be assumed that both the cost and the node-arc coefficients of a support F-arc are sufficient to provide the costs and the coefficients of the contained F-arcs.

Given a F-path

$$p_{xy} = (N_p, E_p)$$

from x to y , for every node $i \in N_p$ a value function $V_p(i)$ is defined as

$$\begin{aligned} V_p(y) &= 0 \\ V_p(t(e)) &= c(e) + \sum_{j \in h(e)} a(e, j) V_p(j), \forall e \in E_p. \end{aligned} \quad (1)$$

$V_p(x)$ is the value of the F-path p_{xy} . Note that the F-path value is an additive function, that is the value of a F-arc tail is the sum of the F-arc cost and a function of values of the nodes of the arc head. Value function was first introduced in [17] for the input-output Leontiev flow problem defined on a particular hypergraph.

Modeling a transit network by a support F-graph

This section shows how to model a transit system by a support F-graph. Such F-graph is obtained by connecting in a specific way several disjoint sub-networks—one for each transit line—together with the ground network. Each line $l \in \mathcal{L}$, where $\mathcal{L}$ is the set of all transit lines, is a closed alternating sequence of nodes representing the line-stops and in-vehicle arcs. In the ground network, the nodes represent geographical locations in the urban area (either stops or zone-centroids), while the arcs represent walking arcs between centroids and/or stops.

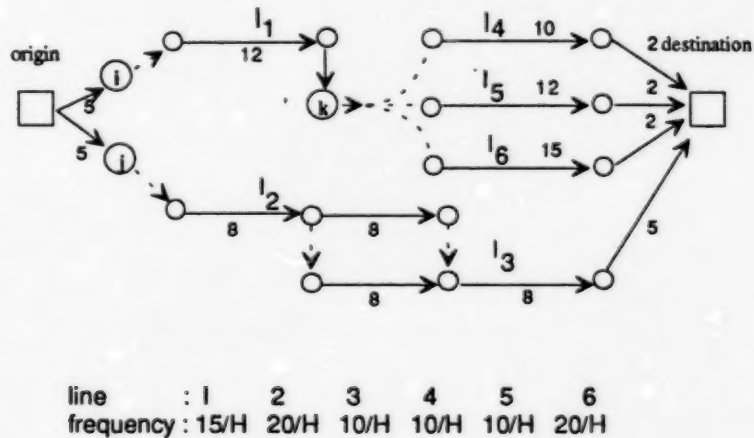


Figure 1: A support F-graph

For each stop-node i in the ground network, let $\mathcal{L}_i$ denote the set of different lines for which i is a stop. This set $\mathcal{L}_i$ may also be specified by the set of nodes corresponding to the line-stops. Node i is connected to the set $\mathcal{L}_i$ by a boarding F-arc $e = (i, \mathcal{L}_i)$. Each line-node $j \in \mathcal{L}_i$ is in turn connected to i by an egress-arc. In the resulting F-graph the set of nodes contains the line-stops, the stop-nodes and the centroids, the set of F-arcs contains boarding F-arcs, in-vehicle arcs and walking and leaving arcs. Figure 1 presents an example of a support F-graph connecting an origin-destination

pair. Consider a passenger waiting at a stop i with ultimate destination d . If this stop-node is served by the set of transit lines $\mathcal{L}_i$, then one possible passenger behavioral model consists in assuming that each passenger has a priori chosen a subset $\mathcal{L}'_i \in \mathcal{L}_i$ —called the attractive set—such that he/she is willing to board the first arriving carrier of this subset to start (or restart) his journey from stop i . On the support F-graph, each attractive set $\mathcal{L}'_i$ is associated with, either a contained, or a support boarding arc $e'_i = (i, h(e'_i))$. Any F-path connecting an origin o to a destination d represents a collection of possible itineraries for a passenger such that every boarding arc describes his/her attractive set at the corresponding stop node. Note that by definition the out-degree of any node of such F-path is always equal to one. Such F-paths is also referred to in previous works as hyperpaths. For example, the F-graph in Figure 1 has nine distinct hyperpaths. This can be seen by noting that the support boarding arc at stop node k has six strictly contained F-arcs, each of these gives rise to a distinct hyperpath passing through node k . Consider a passenger at stop-node i with an associated attractive set $\mathcal{L}'_i$. Let $\xi(e'_i, j)$ denote the probability that the passenger boards the carrier with line-stop $j \in \mathcal{L}'_i$, and $w(e'_i)$ denote the average waiting time at stop i when the attractive set is $\mathcal{L}'_i$. It will be assumed that the probabilities and the average waiting time of any attractive set $\mathcal{L}'_i$ can be locally computed from the data associated with the complete set $\mathcal{L}_i$. In most existing urban models ([18, 21, 6, 30, 16, 23, 15, 32, 27]), the above probabilities and the average waiting time of any attractive set depend only on the frequencies of the lines. For example, under the following assumptions:

- passengers arrive randomly at every stop node, and always board the first arriving carrier of their attractive set;
- all transit lines are statistically independent with given exponential distribution of headway, with mean equal to the inverse of line frequency;

the conditional probabilities $\xi(e'_i, j)$ and the mean waiting time $w(e'_i)$ take on the following specific values

$$\begin{aligned}\Phi(e'_i) &= \sum_{j \in h(e'_i)} \varphi_j, \\ w(e'_i) &= 1/\Phi(e'_i), \\ \xi(e'_i, j) &= \varphi_j/\Phi(e'_i),\end{aligned}\tag{2}$$

where φ_j denotes the frequency of the line corresponding to line-stop $j \in \mathcal{L}_i$. $\Phi(e'_i)$ is often called the *combined frequency* of the attractive set $\mathcal{L}'_i$.

To any egress-arc, walking and in-vehicle arc $e = (i, j)$ is associated a fixed travel time (or generalized cost) $c(e)$ and a coefficient $a(e, j) = 1$, while to each boarding arc $e'_i = (i, h(e'_i))$ there are a waiting time and as many coefficients as the number of nodes in $h(e'_i)$

$$\begin{aligned} c(e'_i) &= w(e'_i) \\ a(e'_i, j) &= \xi(e'_i, j), \forall j \in h(e'_i). \end{aligned} \quad (3)$$

The value $V_p(i)$ of a hyperpath p_{id} connecting node i to destination d in a transit network represents therefore the expected travel time for a passenger to travel from i to d

$$V_p(i) = \begin{cases} c(e) + V_p(j), & \text{if } e = (i, j) \\ w(e'_i) + \sum_{j \in h(e'_i)} \xi(e'_i, j) V_p(j), & \text{if } e'_i \text{ is a boarding arc.} \end{cases} \quad (4)$$

This particular functional form for transit network F-graphs was first introduced in [27]. Within the F-graph framework, the computation of the expected travel times boils down to computing values of F-paths, and this may be carried out by iteratively solving the system of equations (4) in a specific order which corresponds to the inverse topological order of the nodes—from the destination towards the origin node on the acyclic F-path.

We will return to the problem of determining the minimum expected travel time in a F-graph—the shortest hyperpath problem—when *active* F-graphs are discussed.

In passenger assignment models or in origin-destination flow estimation models, one is mainly interested in determining the flow pattern of the transit system given a certain demand. With the above framework, the flow pattern is completely described by the set of active hyperpaths connecting every origin-destination pair, and the associated flows. As a prelude to the investigation of the applicability of multinomial logit models, the next section discusses the decomposition of a given hyperpath flow into arc-flows and the determination of active hyperpaths.

3 Distributing an origin-destination flow onto active hyperpaths

Inferring the arc-flows from a given origin-destination demand f is an essential computational phase in any passenger flow assignment or origin-destination flow estimation model. If this phase required either an explicit

enumeration of all distinct paths of a hyperpath, or that of all active hyperpaths, then the hypergraph framework would be definitely impractical for large-size networks. We will first examine the computation of arc-flows from a hyperpath flow.

Consider a hyperpath $p = p_{od}$ with a given flow f_p . Let Q_p denote the set of all distinct paths with origin o and destination d in hyperpath p_{od} . Recall that p_{od} is an acyclic F-graph. To obviate the explicit enumeration of all paths belonging to Q_p , one should be able to devise an efficient way to compute the arc-shares α_{ep} , for each arc or arc e belonging to hyperpath p , such that the arc flow $x(e, p)$ is given directly by $x(e, p) = \alpha_{ep} f_p$.

These arc-shares are easily obtained if the internal splitting for each arc $e = (t(e), h(e))$ is predefined by a set of positive coefficients

$$\begin{aligned} \sum_{j \in h(e)} a(e, j) &= 1, \\ a(e, j) &\geq 0, \end{aligned} \quad (5)$$

so that for every unit of flow which enters tail node $t(e)$ there is $a(e, j)$ unit which arrives at node $j \in h(e)$. It has been shown in [27] that if the internal splitting of a boarding arc is defined by the coefficients given in (3), then the arc-shares α_{ep} can be obtained by solving the following recursive equations with a breadth-first search of nodes on hyperpath p proceeding from origin o :

$$\alpha_{ep} = \begin{cases} 1, & \text{where } t(e) = o, \\ \sum_{e' \in B(t(e), p)} \alpha_{e'p} a(e', t(e)), & \text{otherwise,} \end{cases} \quad (6)$$

where $B(i, p)$ denotes the backward star of node i in the F-graph of the hyperpath p .

Consider now a specific origin-destination pair (o, d) . Let $\mathcal{P}$ be the set of all hyperpaths connecting this origin-destination pair. We are interested here in the subset of active hyperpaths—hyperpaths which share the total flow of the origin-destination pair. In network flow problems we often distinguish two different approaches in determining the active hyperpaths (or paths in standard network problems) and in splitting the total flow:

- The active hyperpaths are generated iteratively and the flow splitting among members of the current set is updated. Non active hyperpaths are deleted at the end of each iteration. Solution algorithms for the classical problems such as the max flow problem, the minimum cost

flow problem, the equilibrium traffic flow problem (with both symmetric or asymmetric costs, as well as with fixed or variable demand) are based on this approach ([1, 4, 24, 11, 12, 20, 15, 27]).

- The set of all active hyperpaths is defined only once, and the total flow is subdivided among members of the set. For example, determining a blocking flow in a layered network, all or nothing and stochastic traffic assignment procedures belong to this class ([10, 8, 29, 7, 22, 5]). In the case of all or nothing assignment, there is only one active hyperpath, the shortest one, for each origin destination pair.

In general, both approaches rely on shortest paths or shortest hyperpath algorithms ([14, 28]), either to iteratively generate new active hyperpaths, or to define the set of all active hyperpaths.

We will restrict our attention here to the second approach; in particular, to the case when it is possible to predefine the active set without explicitly enumerating all hyperpaths. Such approach requires inevitably that active hyperpaths satisfy certain properties which may be locally verified. The next section shows that the F-graph formed by the set of such active hyperpaths coincides with its support F-graph. In the sequel, such F-graph will be referred to as the *active F-graph*.

The flow splitting can be carried out in different ways. For example, the blocking flow in a layered network can be found by exploring that network with a depth-first search from the origin node ([10]) or by superposing different path flows which saturate the current minimum "throughput" node ([22]). A flow splitting in an active F-graph is called *additive* if the total flow entering any node is subdivided onto the forward arcs in fixed proportions. Additive splitting in a standard active graph may be carried out with a simple breadth-first search procedure if the arc proportions are exogenously given or trivially computed. Performing an additive flow splitting in an active F-graph N_A, E_A is only slightly more involved than in the classical case. Indeed, if for every arc $e = (i, h(e))$ of the active F-graph a local node proportion $b(e, i)$ can be computed, then the splitting of a flow f into arc-flows $x(e)$ and node-flows $y(j)$ is the solution of the following recursive equations:

$$\begin{aligned} y_o &= f, \\ x(e) &= y(t(e)) \cdot b(e, t(e)), \forall e \in E_A, \\ y(j) &= \sum_{e \in B(j)} x(e) \cdot a(e, j), \forall j \in N_A \setminus \{o\}, \end{aligned} \quad (7)$$

where $a(e, j)$ are the internal proportions. These equations may be solved iteratively by exploring the active F-graph in topological order from the origin o toward the destination d . The interesting question is how to determine the active F-graph and the local node-proportions $b(e, i)$.

As mentioned earlier, to define the active set without explicitly enumerating all possible hyperpaths some restriction must be imposed on active hyperpaths. We define an active hyperpath (N_p, E_p) as a hyperpath in which all arcs satisfy the property that the minimum value of the tail node is strictly greater than that of the head nodes:

$$V^*(t(e)) > V^*(j), \forall j \in h(e); \forall e \in E_p. \quad (8)$$

Note that the minimum value at node i , $V^*(i)$, represents the minimum expected travel time from i to destination d for the present application, thus an active arc may be perceived as an arc which brings the passenger closer to the destination. This extends the definition of active arcs, first introduced in [8].

One crucial implication of this definition for the hyperpath framework using the waiting time model (2) is

Proposition 3.1 *At any stop node i served by k competing transit lines only one boarding arc is active among the $2^k - 1$ possible arcs.*

This result can be seen as follows. Consider a stop node i and its support boarding arc $e = (i, h(e))$. Assume that the minimum values $V^*(j), \forall j \in h(e)$ with respect to destination d are given. Such values may be computed, for each destination d , with one of the polynomial shortest hyperpaths algorithms developed in [28]. Let $\bar{e} = (i, h(\bar{e}))$ be the boarding arc with minimal value $V^*(i)$ among the $2^k - 1$ contained arcs:

$$V^*(t(\bar{e})) \leq V^*(t(e')), \quad \forall e' \in \{(i, h(e')) \mid h(e') \subseteq h(e)\}.$$

As shown in [26], the boarding arc $\bar{e} = (i, h(\bar{e}))$ has the following properties:

$$\begin{aligned} V^*(t(\bar{e})) &> V^*(j), \quad \forall j \in h(\bar{e}), \\ V^*(t(\bar{e})) &\leq V^*(j), \quad \forall j \in h(e) \setminus h(\bar{e}). \end{aligned} \quad (9)$$

Thus, the boarding arc $\bar{e} = (i, h(\bar{e}))$ with the minimal head set cardinality $|h(\bar{e})|$ is clearly the unique active boarding arc at stop node i . Using the above property (9), the active boarding arc at any stop node may be determined with the following greedy procedure:

```

Procedure Active_arc (  $V^*, e, h(\bar{e})$  )
{
    sort the nodes  $\ell \in h(e)$  in increasing order of values  $V^*(\ell)$ ;
     $\Phi := \varphi_{\ell_1}$ ;
     $A := \ell_1$ ;
     $w := 1/\Phi$ ;
     $C := w + V^*(\ell_1)$ ;
     $i := 2$ ;
    while  $C > V^*(\ell_i)$  and  $i \leq |h(e)|$  do {
         $A := A \cup \{\ell_i\}$ ;
         $\Phi := \Phi + \varphi_{\ell_i}$ ;
         $w := 1/\Phi$ ;
         $C := C - \frac{(C - V^*(\ell_i)) \cdot \varphi_{\ell_i}}{\Phi}$ ;
         $i := i + 1$ 
    }
     $h(\bar{e}) := A; V^*(t(\bar{e})) := C$ 
}

```

Applying the above procedure to the boarding arc at stop node k in Figure 1 leads to the active F-graph illustrated in Figure 2. The number of

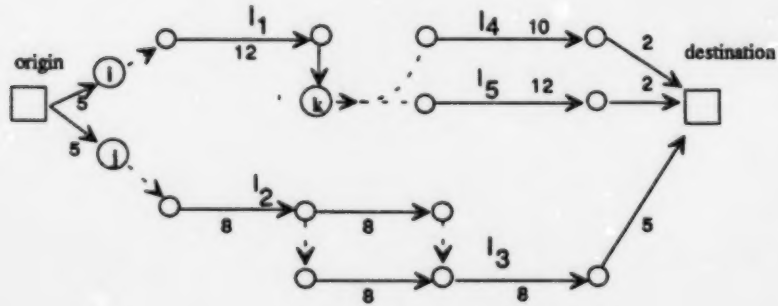


Figure 2: An active F-graph

hyperpaths is now reduced to three (Figure 3). Generating the active graph

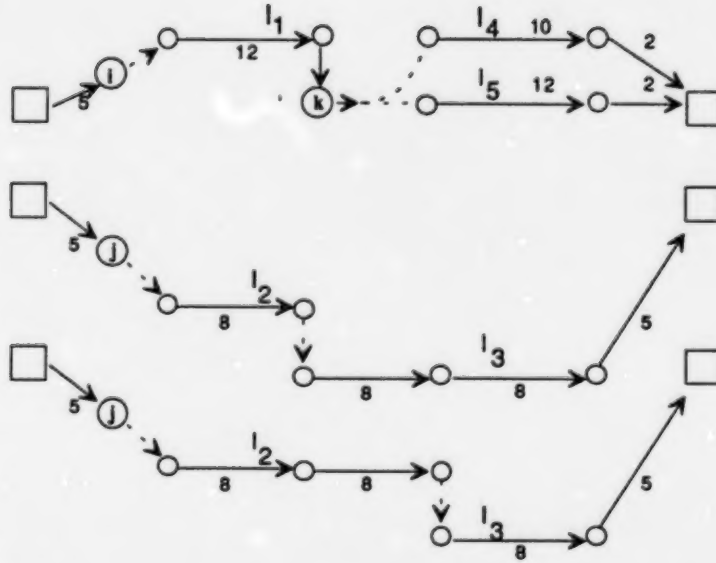


Figure 3: Active hyperpaths

is therefore accomplished in polynomial time by executing a shortest hyperpath algorithm. With a label setting approach ([9]), the above procedure may be directly embedded in the label revision phase of the algorithm. The following is an implementation of this approach and determines a shortest hyper-tree rooted at a given destination d . Note that in contrast to the ordinary graph applications, shortest hyperpaths can only be computed from the destination. At completion the algorithm returns, for a given destination d , the optimal node values $V^*(j)$ and the head set $h(\bar{e}) = h^*(e)$ of every active boarding arc. The shortest hyper-tree rooted at destination d may be retraced with the successor function $S(j)$.

Procedure *Shortest_hyperpath* ($\mathcal{H}, d, h^*, V^*, S$)

```
{ (initialization)
  for each  $j \in N$  do  $V^*(j) := \infty$ ;
  for each  $e \in E$  do  $\{C^*(e) := \infty; h^*(e) := \emptyset; \Phi^*(e) := 0\}$ 
```

```

A := {d}; V*(d) := 0;
Until A = ∅ repeat {
  (node selection)
  j = arg min{V*(ℓ) | ℓ ∈ A}; A := A - {j};
  (label revision)
  for each e ∈ B(j) such that V*(j) < V*(t(e)) do {
    if Φ*(e) = 0
      then C*(e) := 1/φj + V*(j)
      else C*(e) := C*(e) -  $\frac{(C^*(e) - V^*(j))\varphi_j}{\Phi^*(e)}$ 
    end-if
    h*(e) := h*(e) ∪ {j}; Φ*(e) := Φ*(e) + φj;
    if C*(e) < V*(t(e))
      then { V*(t(e)) := C*(e); S(t(e)) := e;
            if t(e) ∉ A then A := A ∪ {t(e)}
            }
    end-if
  }
}
}.

```

In the above algorithm the number of node selections is bounded by $|N|$ and the number of times that the value of a node can be updated is bounded by the size of the support F-graph $\text{size}(\mathcal{H}) \leq |E|\ell$, where ℓ is the cardinality of the largest active boarding arc. If the candidate list A is implemented as an unordered list, we obtain a version which runs in $O(|N|^2 + |E|\ell)$ time. Otherwise, if a binary heap is used for A , we obtain a $O(|E|\ell \log |N|)$ variant.

4 Multinomial logit models

The standard logit traffic assignment model was first introduced in [8] and provides an elegant computational traffic assignment procedure for road networks. Albeit its well known limitations, the logit traffic assignment model still represents a viable alternative to either the probit model or the deterministic equilibrium model, due to its moderate demand in network data and implementation effort. There are though numerous pitfalls in considering the application of the standard logit distribution model to the transit

networks described by support F-graphs. Indeed, assume that the probability of using any hyperpath $p \in P(o, d)$ is given by the classical logit rule

$$\Pr(p) = \frac{\exp(\theta V_p)}{\sum_{l \in P(o, d)} \exp(\theta V_l)}, \quad \theta < 0. \quad (10)$$

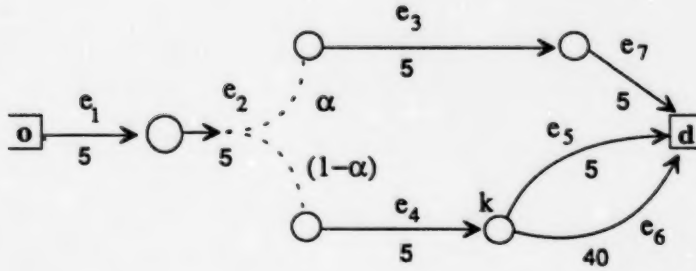


Figure 4: A case of two active hyperpaths

With a little thought it is easy to see that an implicit enumeration of the active hyperpaths cannot be implemented with a naive application of the logit model. Consider for instance the hypergraph in Figure 4. The two hyperpaths (p_1, p_2) connecting origin o and destination d differ only by the segments e_5 and e_6 . The values of these two hyperpaths are

$$\begin{aligned} V(p_1) &= c(e_1) + c(e_2) + \alpha(c(e_3) + c(e_7)) \\ &\quad (1 - \alpha)(c(e_4) + c(e_5)), \\ V(p_2) &= c(e_1) + c(e_2) + \alpha(c(e_3) + c(e_7)) \\ &\quad (1 - \alpha)(c(e_4) + c(e_6)), \end{aligned}$$

where α is the probability of boarding the carriers corresponding to line segment e_3 . At node k , the conditional probability of choosing arc e_5 must be equal to that of choosing hyperpath p_1 . However, this probability cannot be calculated solely with information gathered from the destination d back to node k , since the value (cost) of a hyperpath is not necessarily an additive function of the costs of the embedded F-arcs.

In addition to losing the algorithmic advantages which is one of the main reasons for adopting the logit model (10), the hyperpath framework seems to inflate the inherent limitations of the standard logit model. Consider again the active F-graph in Figure 3. According to the logit rule, the conditional probability of choosing arc e_5 at node k is

$$\Pr(e_5) = (1 - \alpha) \Pr(p_1),$$

and that of e_6 is

$$\Pr(e_6) = (1 - \alpha) \Pr(p_2).$$

With $\alpha = 0.80$ and $\theta = -0.10$, the probability of choosing e_5 is only twice that of e_6 (0.133 and 0.067, respectively). Considering that the cost of segment e_6 is eight time larger than that of e_5 , these results will undoubtedly lead to an aberrant flow distribution. It is interesting to note that if the logit rule is applied to segments e_5 and e_6 , independently from the hyperpath p_1 and p_2 , then the probability of choosing e_6 decreases to a more reasonable value of 0.029. In retrospect, it is not surprising to find such aberration, since in the preceding example the two alternatives are strongly correlated. More precisely, the presence of the common line segment e_3 with an associated probability of 0.80 dilutes somewhat the impact of the discrepancy in the travel costs between alternative e_5 and e_6 .

It is also worth noting that a direct application of the logit model to the active paths (in contrast to active hyperpaths) will obviously destroy the internal splitting rule embedded in the boarding F-arc framework.

In view of the previous analysis, a passenger routing decision mechanism based on a logit model should definitively be implemented with a sequential decision rule. In such approach the choice at any given point on the network is assumed to be unaffected by the attributes of the path used to reach that point—an approach reminiscent of the decision rule which pervades the classical dynamic programming framework. The rationale for adopting a sequential approach, in contrast to a simultaneous choice approach, is that the actual passenger's itinerary does not depend solely on his choice but also on the realization of some external random events such as the arrival at a stop node of the first carrier of his choice set. One possible alternative is to apply the *composite* cost concept embedded in the hierarchical logit model which is commonly used for the estimation of choice probabilities in travel demand applications. It seems to retain the computational simplicity of the simple logit model and also can account for certain correlation among

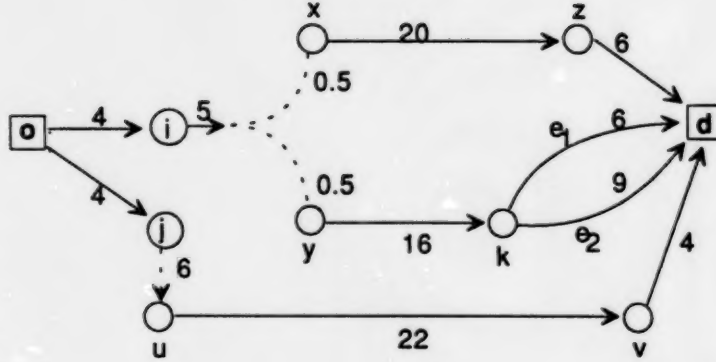


Figure 5: Logit distribution of flow on an active hypergraph

alternatives. For a detailed account of the hierarchical logit model the reader is referred to [33, 31, 19].

To highlight a sequential approach using a "log sum" composite cost, consider the example given in Figure 5. The three active hyperpaths, illustrated on Figure 6, have the following values

$$\begin{aligned} V(p_1) &= 4 + 5 + 0.5(20 + 6) + 0.5(16 + 6) = 33, \\ V(p_2) &= 4 + 5 + 0.5(20 + 6) + 0.5(16 + 9) = 34.5, \\ V(p_3) &= 4 + 6 + 22 + 4 = 36. \end{aligned}$$

Assume that $\theta = -0.10$, then the standard logit model leads to the following probabilities: $\Pr(p_1) = 0.38$, $\Pr(p_2) = 0.33$ and $\Pr(p_3) = 0.29$, according to equation (10).

In a sequential approach, the first two hyperpaths are replaced by a composite hyperpath p_{12} , in which, from a conceptual point of view, the two alternatives e_1 and e_2 are collapsed into a composite alternative $\bar{e}$ (Figure 7). The value of this latter is defined as

$$c(\bar{e}) = (1/\theta) \ln\{\exp[\theta c(e_1)] + \exp[\theta c(e_2)]\}. \quad (11)$$

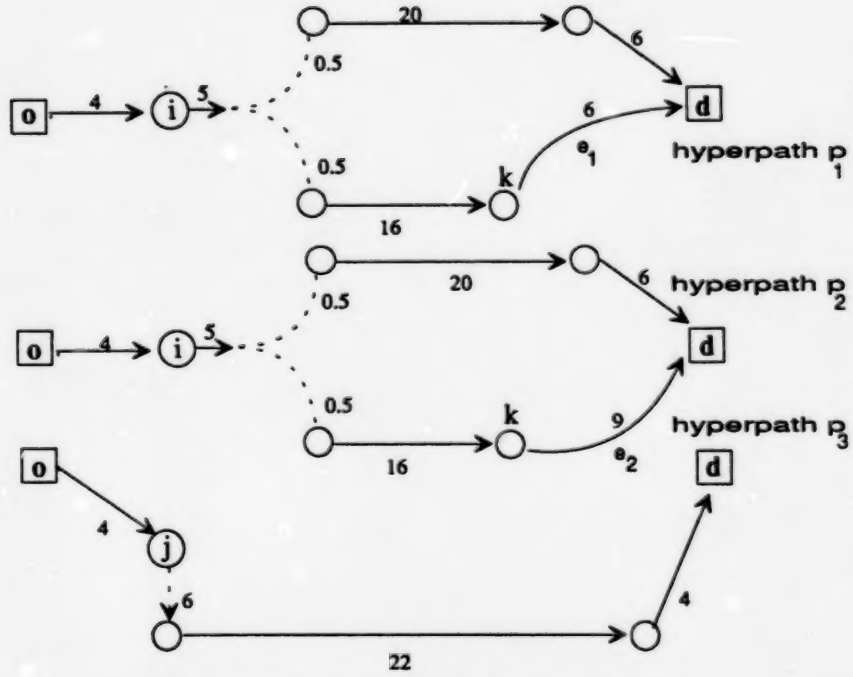


Figure 6: An enumeration of active hyperpaths

The value of the composite hyperpath can now be calculated as follow

$$V(p_{12}) = 4 + 5 + 0.5(20 + 6) + 0.5(16 + c(\bar{e})).$$

This leads to the exponential expression

$$\exp[\theta V(p_{12})] = \exp[\theta(4 + 5)] \cdot (\exp[\theta(20)] \cdot \exp[\theta(6)]^{0.5} \cdot (\exp[\theta(16)] \cdot (\exp[\theta(6)] + \exp[\theta(9)])^{0.5}), \quad (12)$$

and the probability of choosing this composite hyperpath is given, as before, by the multinomial logit ratio (10)

$$\Pr(p_{12}) = \frac{\exp[\theta V(p_{12})]}{\exp[\theta V(p_{12})] + \exp[\theta V(p_3)]}. \quad (13)$$

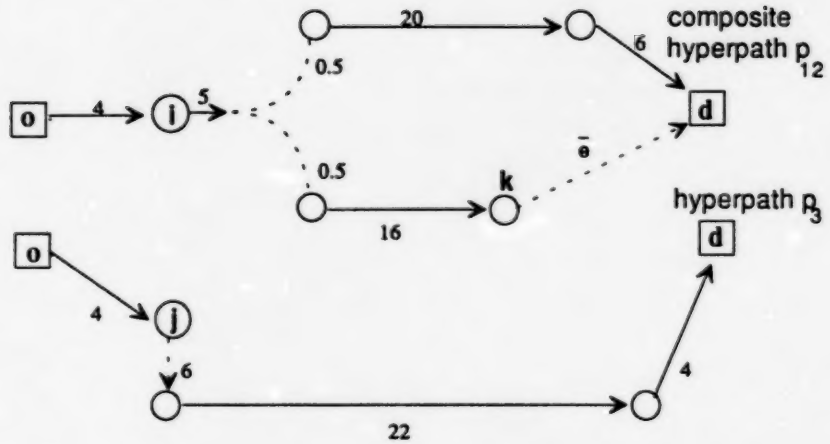


Figure 7: Composite hyperpaths

A closer look at equation (12) reveals that one can compute the above probability without an explicit generation of active hyperpaths. Indeed, consider the following recursive system of equations:

$$W_i = \begin{cases} 1, & \text{if } i = d, \\ \exp[\theta c(e)] \prod_{j \in h(e)} W_j^{a(e,j)}, & \text{if } i \text{ is a stop node,} \\ \sum_{j \in F(i)} \exp[\theta c(e)] \cdot W_j, & \text{otherwise.} \end{cases} \quad (14)$$

This system can be solved iteratively in the inverse topological order from the destination d toward the origin o on the active F-graph. For the current

example, the following values are obtained:

$$\begin{aligned}
W_k &= \exp(-0.6) \cdot 1 + \exp(-0.9) \cdot 1 &= 0.955 \\
W_x &= \exp(-0.6) \cdot 1 &= 0.549 \\
W_v &= \exp(-0.4) \cdot 1 &= 0.670 \\
W_z &= \exp(-2.0) \cdot W_x &= 0.074 \\
W_y &= \exp(-1.6) \cdot W_k &= 0.193 \\
W_u &= \exp(-2.2) \cdot W_v &= 0.074 \\
W_i &= \exp(-0.5) \cdot W_x^{0.5} \cdot W_y^{0.5} &= 0.073 \\
W_j &= \exp(-0.6) \cdot W_u &= 0.041 \\
W_o &= \exp(-0.4) \cdot W_i + \exp(-0.4) \cdot W_j &= 0.076 \\
\Pr(p_{12}) &= \exp(-0.4) \cdot W_i / W_o &= 0.640.
\end{aligned}$$

The probability of choosing the composite hyperpath is significantly less than the combined probability of choosing either hyperpath p_1 or p_2 , which is the expected result.

5 Conclusion

We now collect all the results obtained so far in an algorithmic procedure which can be used to produce an assignment of passenger demands from all origins to a given destination according to a hierarchical multinomial logit distribution with a single parameter θ . By cycling through all destinations a complete traffic pattern may be obtained. In a first phase, the corresponding active F-graph must be determined with a shortest hyperpath algorithm such as the procedure described in the previous section. Active boarding arcs are given by $h^*(e)$ and all other inactive arcs are deleted from the F-graph by checking inequality (8). We will now assume that $h^*(e)$ is defined for every arc of the F-graph. In particular,

$$\begin{aligned}
|h^*(e)| &\geq 1, \\
c^*(e) &= 1/\Phi^*(e), \\
a^*(e, j) &= \varphi_j/\Phi^*(e),
\end{aligned}$$

for an active boarding arc and

$$\begin{aligned}
|h^*(e)| &= 1, \\
c^*(e) &= c(e), \\
a^*(e, j) &= 1,
\end{aligned}$$

for all other active arcs and $|h^*(e)| = 0$, otherwise. The second phase is divided in two passes. In the first pass, the active F-graph is visited in the inverse topological order starting from the destination d . This pass solves the recursive system of equations (14) and simultaneously determines the local node proportion $b(e, j)$ (equation (7)). In the second pass, this time in a direct topological order from the origins towards destination d , the passenger demands are allocated to active F-arcs. The (direct and inverse) topological ordering is inferred from the shortest hyperpath algorithm used. For instance, if algorithm *Shortest-hyperpath*($\mathcal{H}, d$) is implemented then the order in which nodes are removed from the list of candidates A corresponds to the inverse topological order. Let T denote the list of nodes in inverse topological order. To obtain this list simply insert an additional statement in the shortest hyperpath procedure as follows

```

      :
Until   $A = \emptyset$   repeat {
  (node selection)
   $j = \arg \min \{V^*(\ell) \mid \ell \in A\}; \quad A := A - \{j\};$ 
   $T := T \cup \{j\};$  (insert at the tail )
  (label revision)
      :
}.

```

The first backward pass is realized with the following procedure *Local-Splitting*. In this procedure, if $i \in T$ is a stop-node, then $(i, h^*(e)) \in F(i)$ is the unique arc incident out of i .

```

Procedure Local-Splitting ( $d, \mathcal{H}, h^*, c^*, a^*, T, b$ )
{
  (initialization)
  for each  $i \in N$  do  $W(i) := 0$ ;
  for each  $e \in E$  do  $b(e, t(e)) := 0$ ;
   $W(d) := 1$ ;
  (visit the F-graph in inverse topological order)
  for each  $i \in T \setminus \{d\}$  do {
    if  $i$  is a stop-node
      then  $\{W(i) := \exp(\theta c^*(e)) \prod_{j \in h^*(e)} W(j)^{a^*(e,j)};$ 
            $b(e, i) := 1\}$ 

```

```

    else {  $W(i) := \sum_{e \in F(i) \mid |h^*(e)| \neq 0} \exp(\theta c^*(e)) \cdot W(h^*(e));$ 
            $b(e, i) := \exp(\theta c^*(e)) \cdot W(h^*(e)) / W(i)$ 
    }
  end-if
}
}.

```

Procedure *Local-Splitting* runs in $O(|E|\ell)$ time. Finally, the last step is devoted to the flow distribution according to the local and the internal splitting coefficients. The following procedure *Flow-Distribution* algorithm solves system (7) simultaneously for all origins to one specific destination d . $f(i, d)$ is the demand from origin i to destination d , $f(i, d) > 0$ for all active origin i , and $f(i, d) = 0$, otherwise. The resulting traffic pattern is described by the node-flows $y(i)$ and the arc-flows $x(e)$. T^{-1} denotes the list of nodes in the direct topological order (nodes are simply fetched from the tail of T).

```

Procedure Flow-Distribution( $f, d, \mathcal{H}, h^*, T, a^*, b, x, y$ )
{
  (initialization)
  for each  $i \in N$  do  $y(i) := f(i, d)$ ;
  for each  $e \in E$  do  $x(e) := 0$ ;
  (visit the F-graph in direct topological order)
  for each  $i \in T^{-1} \setminus \{d\}$  do {
    for each  $e \in F(i)$  such that  $|h^*(e)| = 1$  do{
       $x(e) := y(i) \cdot b(e, i)$ ;
      for each  $j \in h^*(e)$  do
         $y(j) := y(j) + y(i) \cdot a^*(e, i)$ 
      }
    }
  }
}
}.

```

Procedure *Flow-Distribution* also runs in $O(|E|\ell)$ time. The following procedure *Logit-Distribution* grafts together the three previous steps

```

Procedure Logit-Distribution( $f, d, H, v$ )
{ call Shortest_Hypergraph ( $\mathcal{H}, d, h^*, V^*, T$ );
  call Local-Splitting ( $d, \mathcal{H}, h^*, c^*, a^*, T, b$ );
  call Flow-Distribution( $f, d, \mathcal{H}, h^*, T, a^*, b, x, y$ );
}
}.

```

The complexity of *Logit-Distribution* depends on the version of shortest hyperpaths adopted and indeed has the same complexity than this latter.

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